

Thermodynamic duality between RN black hole and 2D dilaton gravity

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Abstract

All thermodynamic quantities of the Reissner-Nordström (RN) black hole can be obtained from the dilaton and its potential of two dimensional (2D) dilaton gravity. The dual relations of four thermodynamic laws are also established. Furthermore, the near-horizon thermodynamics of the extremal RN black hole is completely described by the Jackiw-Teitelboim theory which is obtained by perturbing around the AdS₂-horizon.

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I. INTRODUCTION

Since the pioneering work of Bekenstein [1, 2, 3] and Hawking [4], which proved that the entropy of black hole is proportional to the area of its horizon in the early 1970s, the research of the black hole thermodynamics has greatly improved. Especially, the proof of the Hawking radiation stimulated the enthusiasm for studying the thermal property of the black hole. It was found that if the surface gravity of the black hole is considered to be the temperature and the outer horizon area is considered to be the entropy, four laws of the black hole thermodynamics, which correspond with the four laws of the elementary thermodynamics, has been established [5].

On the other hand, 2D dilaton gravity has been used in various situations as an effective description of 4D gravity after a black hole in string theory has appeared [6, 7]. Hawking radiation and thermodynamics of this black hole has been analyzed by several authors [8, 9, 10, 11]. Another 2D theories, which were originated from the Jackiw-Teitelboim (JT) theory [12, 13], have been also studied [14, 15, 16]. Although in this theory the curvature is constant and negative, it has a black hole solution, which implies a non-trivial causal structure and in turn generates interesting non-trivial thermodynamics [17, 18, 19, 20, 21, 22]. Moreover, Fabbri *et. al.* [23] partially demonstrated the duality of the thermodynamics between near-extremal RN black hole and the JT theory because they considered temperature and entropy only. Recently, Grumiller and McNees [24] discussed black hole thermodynamics in the 2D dilaton gravity (for a review see [25]).

In this Letter, we completely describe the thermodynamic duality between the RN black hole and the 2D dilaton gravity. The key ingredient is to fully use the dilaton potential induced by the dimensional reduction and the conformal transformation. Moreover, we also show the thermodynamic duality between near-extremal RN black hole and the JT theory.

II. RN THERMODYNAMICS

Consider the RN black hole, whose metric is given by

$$ds_{RN}^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (1)$$

with $U(r) = 1 - 2M/r + Q^2/r^2$. Here, M and Q are the mass and the electric charge of the RN black hole, respectively. Then, the inner (r_-) and the outer (r_+) horizons are obtained

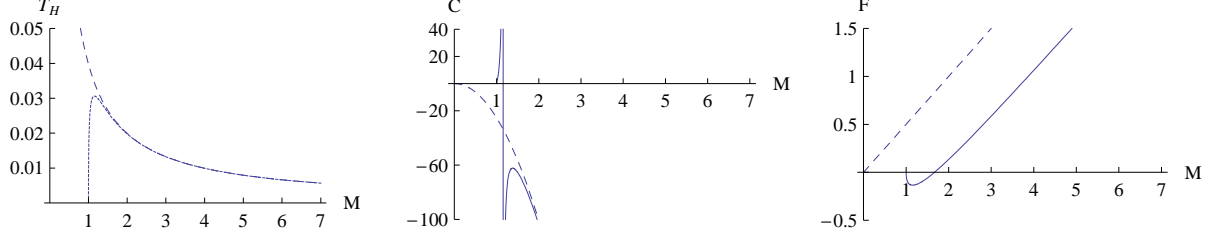


FIG. 1: Thermodynamic quantities of temperature, heat capacity, and free energy as function of mass M with fixed $Q = 1$. The solid curves represent the RN black hole, while the dashed curves denote the Schwarzschild black hole with $Q = 0$.

as $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$, which satisfy $U(r_{\pm}) = 0$. For $M_e = Q$, we have an extremal RN black hole at $r_e = Q$. In this work we consider the case of fixed charge Q for simplicity [26]. The other case of the fixed potential $\Phi = Q/r_+$ will have parallel with the fixed charge case.

For the RN black hole, the relevant thermodynamic quantities are given by the Bekenstein-Hawking entropy and Hawking temperature

$$S_{BH}(M, Q) = \pi \left(M + \sqrt{M^2 - Q^2} \right)^2, \quad (2)$$

$$T_H(M, Q) = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2}. \quad (3)$$

Then, using the Eqs. (2) and (3), the heat capacity $C = (dM/dT_H)_Q$ and Helmholtz free energy F are obtained to be

$$C(M, Q) = \frac{2\pi\sqrt{M^2 - Q^2}(M + \sqrt{M^2 - Q^2})^3}{2Q^2 - M(M + \sqrt{M^2 - Q^2})}, \quad (4)$$

$$F(M, Q) = E - T_H S_{BH} = E - \frac{1}{2}\sqrt{M^2 - Q^2} \quad (5)$$

with $E = M - Q$. We note here that one has to use the extremal black hole as background for fixed charge ensemble [26].

As is shown in Fig. 1, the features of thermodynamic quantities are as follows: First of all, the whole region splits into the near-horizon phase of $Q < M < M_m$ with $M_m = \frac{2}{\sqrt{3}}Q$ and Schwarzschild phase of $M > M_m$. i) The temperature is zero ($T_H = 0$) at $M = Q$, maximum $T_m = \frac{1}{6\sqrt{3}\pi Q}$ at $M = M_m$, and for $M > M_m$, it shows the Schwarzschild behavior. ii) The heat capacity C determines thermodynamic stability. For $Q < M < M_m$, it is stable because of $C > 0$, while for $M > M_m$, it is unstable ($C < 0$) as is shown in the Schwarzschild

case. Here, we have $C = 0$ at $M = Q$, and importantly C blows up at $M = M_m$. iii) The free energy is an important quantity to determine the presumed phase transition. The free energy is always negative and it is decreasing ($E < T_H S_{BH}$) for the near-horizon phase and increasing ($E > T_H S_{BH}$) for the Schwarzschild phase. It takes the minimum value $F_m = \frac{Q(\sqrt{3}-2)}{2}$ at $M = M_m$ and $F = 0$ at $M = Q$. Here, we classify the extremal RN black hole by the conditions of $T_H = 0$, $C = 0$, $S_{BH} = \pi Q^2$, $F = 0$.

III. 2D DILATON GRAVITY INDUCED BY DIMENSIONAL REDUCTION

We start with the four-dimensional action whose solution is given by Eq.(1) as

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R - F_{\mu\nu} F^{\mu\nu}] \quad (6)$$

with $F_{\mu\nu} F^{\mu\nu} = 2Q^2/r^2$. After the dimensional reduction by integrating over S^2 and eliminating the kinetic term by using the conformal transformation

$$\bar{g}_{\mu\nu} = \sqrt{\phi} g_{\mu\nu}, \quad \phi = \frac{r^2}{4}, \quad (7)$$

the reparameterized action is obtained as

$$\bar{I}^{(2)} = \int d^2x \sqrt{-\bar{g}} [\phi \bar{R}_2 + V(\phi)], \quad (8)$$

where the 2D Ricci scalar $\bar{R}_2 = -\frac{U''}{\sqrt{\phi}}$ and the dilaton potential is given by

$$V(\phi) = \frac{1}{2\sqrt{\phi}} - \frac{Q^2}{8\phi^{3/2}}. \quad (9)$$

This is an effective 2D dilaton gravity with $G_2 = 1/2$ [12]. The two equations of motion are

$$\nabla^2 \phi = V(\phi), \quad (10)$$

$$\bar{R}_2 = -V'(\phi), \quad (11)$$

where the derivative of V is given by

$$V'(\phi) = -\frac{1}{4\phi^{3/2}} + \frac{3Q^2}{16\phi^{5/2}}. \quad (12)$$

Then, we obtain the general solution to Eqs. (10) and (11) by choosing a conformal gauge of $\bar{g}_{tx} = 0$ [27, 28, 29] as

$$\frac{d\phi}{dx} = 2(J(\phi) - \mathcal{C}), \quad (13)$$

$$ds_{2D}^2 = -(J(\phi) - \mathcal{C})dt^2 + \frac{dx^2}{J(\phi) - \mathcal{C}}, \quad (14)$$

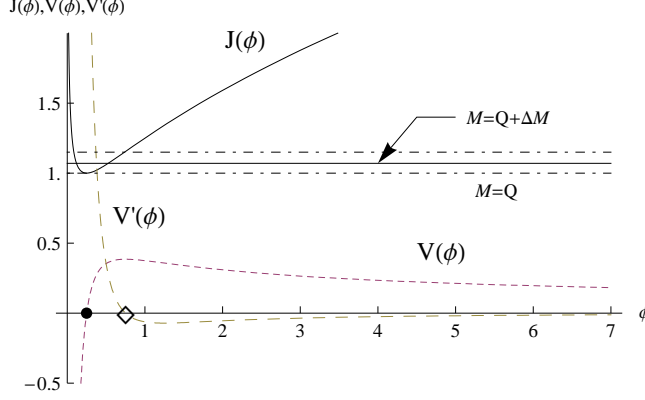


FIG. 2: Three graphs for the 2D dilaton gravity with $Q = 1$. The solid curve describes $J(\phi)$, the dotted curve gives $V(\phi)$, the dashed curve denotes $V'(\phi)$, and the near-horizon region is between the dot-dashed lines. $\bullet(\diamond)$ represent $\phi = \phi_0(\phi_m)$. The horizontal line at $M = Q + \Delta M$ is for the mass perturbation of the Eq. (23).

where $J(\phi)$ is the integration of V given by

$$J(\phi) = \int^{\phi} V(\tilde{\phi}) d\tilde{\phi} = \sqrt{\phi} + \frac{Q^2}{4\sqrt{\phi}}. \quad (15)$$

Here, \mathcal{C} is a coordinate-invariant constant of integration, which is identified with the mass M of the RN black hole. We note here the important connection between $J(\phi)$ and the metric function $U(r(\phi))$ with $r = 2\sqrt{\phi}$: $\sqrt{\phi} U(\phi) = J(\phi) - M$. For $\phi = \text{const.}$, we have the AdS_2 -horizon which satisfies $J = M$ that is equivalent to $U(r_{\pm}) = 0$.

As is shown in Fig. 2, we observe two important points: one (\bullet) is $\phi = \phi_0 = Q^2/4$, where $J = Q$, $J' = V = 0$, $J'' = V' = 4/Q^3$. Another (\diamond) is $\phi = \phi_m = 3Q^2/4$, where $J = \frac{2}{\sqrt{3}}Q$, $J' = V = \frac{2}{3\sqrt{3}}Q$, $J'' = V' = 0$. The former shows the extremal configuration, while the latter indicates the unstable maximum point. All thermodynamic quantities can be explicitly expressed in terms of the dilaton ϕ , the dilaton potential $V(\phi)$, its integration $J(\phi)$, and its derivative $V'(\phi)$ as

$$\begin{aligned} S_{BH}(\phi) &= 4\pi\phi, \quad T_H(\phi) = \frac{V(\phi)}{4\pi}, \quad C(\phi) = 4\pi \frac{V(\phi)}{V'(\phi)} \\ F(\phi) &= J(\phi) - J(\phi_0) - \phi V(\phi). \end{aligned} \quad (16)$$

These are our main results. In Fig. 3, we have the corresponding dual graphs, which are nearly the same as in Fig. 1. For $\phi_0 < \phi < \phi_m$, we have the JT phase, whereas for $\phi > \phi_m$, we have the Schwarzschild phase. At the extremal point $M = Q$, we have

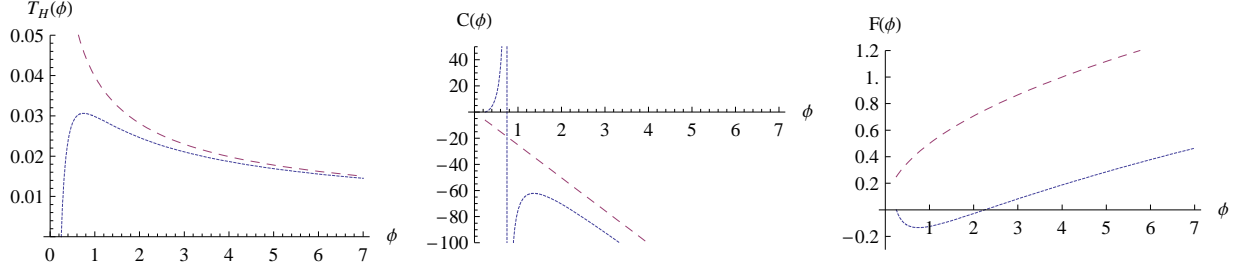


FIG. 3: Graphs for RN thermodynamic quantities expressed in terms of J, V, V' as functions of ϕ . Here ϕ plays the role of the entropy. The solid curves represent the RN black hole with $Q = 1$, while the dashed curves denote the Schwarzschild black hole with $Q = 0$.

	0th law	1st law	2nd law	3rd law
RN black hole	$T_H = \kappa/2\pi$	$dM = T_H dS_{BH} + (\Phi - \Phi_e)dQ$	$\Delta S_{BH} \geq 0$	$T_H \rightarrow 0$
2D dilaton gravity	$T_H = V/4\pi$	$dJ = Vd\phi + (\frac{Q}{2\sqrt{\phi}} - \frac{Q}{2\sqrt{\phi_0}})dQ$	$\Delta\phi \geq 0$	$V \rightarrow 0$

TABLE I: Dual relations of four thermodynamic laws. Here the potential and its extremal value are given by $\Phi = Q/r_+$ and $\Phi_e = Q/r_e = 1$ [26].

$T_H = 0$, $C = 0$, $F = 0$, which are determined by $V(\phi_0) = 0$. On the other hand, at the maximum point $M = M_m$, one has $T_H = T_m$, $C = \pm\infty$, $F = F_m$, which are fixed by $V'(\phi_m) = 0$.

IV. FOUR THERMODYNAMIC LAWS

As is shown in Table I, we confirm that the dual relation of four thermodynamic laws can be easily realized by the 2D dilaton and its potential. The first law is given by the definition of J : $dJ = Vd\phi$. Concerning on the third law, it has a rather different status from the others. It was proved that an infinite amount of time is required for the near-extremal RN black hole to decay to extremality at $T_H = 0$ [23]. Similarly, we require that it take an infinite time to arrive at $V(\phi_0) = 0$ from $V(\phi_0) \neq 0$ located at the near-horizon phase.

V. DUALITY BETWEEN NEAR-EXTREMAL RN BLACK HOLE AND JT THEORY

It is very interesting to explore the near-horizon thermodynamics to the extremal RN black hole. This could be obtained by inserting $\Delta M = M - Q$ into Eqs. (2)-(5) to leading order in $\sqrt{\Delta M}$. Then, all near-horizon thermodynamic quantities are given by

$$S_{NH} = 2\pi Q \sqrt{2Q\Delta M}, \quad T_{NH} = \frac{\sqrt{2Q\Delta M}}{2\pi Q^2}, \quad (17)$$

$$C_{NH} = 2\pi Q \sqrt{2Q\Delta M}, \quad F_{NH} = -\frac{\sqrt{2Q\Delta M}}{2} \quad (18)$$

with the definition of $f_{NH} \equiv f(M, Q) - f(Q, Q)$. In general, the near-horizon quantities do not satisfy the first law of thermodynamics. Instead, we find $2\Delta M = T_{NH}S_{NH}$ and $S_{NH} = C_{NH}$, which show the same behavior as in the non-rotating BTZ black hole [30].

In order to find the AdS_2 gravity of the JT theory, we consider the perturbation around the AdS_2 -horizon as

$$J(\phi) \simeq Q + \frac{V'(\phi_0)}{2}\varphi^2, \quad (19)$$

$$M \simeq Q[1 + k\alpha^2] \equiv Q + \Delta M \quad (20)$$

with $\varphi = \phi - \phi_0$, $J'(\phi_0) = V(\phi_0) = 0$, and $J''(\phi_0) = V'(\phi_0)$. Introducing the new coordinates

$$\tilde{t} = \alpha t, \quad \tilde{x} = \frac{x - x_e}{\alpha}, \quad (21)$$

then the perturbed metric function with the perturbed dilaton $\varphi = \alpha\tilde{x}$ is given by

$$ds_{JT}^2 = -\left[\frac{V'(\phi_0)}{2}\tilde{x}^2 - kQ\right]d\tilde{t}^2 + \frac{d\tilde{x}^2}{\left[\frac{V'(\phi_0)}{2}\tilde{x}^2 - kQ\right]}, \quad (22)$$

which shows a locally AdS_2 spacetime. If $k = 0$, it is a globally AdS_2 spacetime. Moreover, the mass deviation ΔM is the conserved parameter of the JT theory [28].

Now, we are in a position to describe the near-horizon thermodynamics from the JT theory. From the null condition of the metric function in Eq. (22), we have the positive root

$$\tilde{x}_+ = \sqrt{\frac{2kQ}{V'(\phi_0)}} \rightarrow \varphi_+ = \frac{Q}{2}\sqrt{2Q\Delta M}. \quad (23)$$

Then, the JT thermodynamics can be obtained by perturbing the thermodynamic quantities in Eq. (16) around $\phi = \phi_0$ as

$$S_{JT} = 4\pi\varphi_+, \quad T_{JT} = \frac{V'(\phi_0)\varphi_+}{4\pi}, \quad (24)$$

$$C_{JT} = 4\pi\varphi_+, \quad F_{JT} = -\frac{\varphi_+}{Q}. \quad (25)$$

As a result, we confirm that $f_{JT} = f_{NH}$. This means that the near-horizon (AdS_2) thermodynamics of the extremal RN black hole can be completely described by the JT theory. We note that the equality between the near-horizon thermodynamics and JT theory was partially proved for the entropy and temperature up to now [23]. In this work, we have completely described the dual relations for all thermodynamic quantities including the heat capacity and free energy. Furthermore, assuming asymptotic symmetry at boundary with a periodicity of $2\pi\beta$ in \tilde{t} [31], we could also show that the JT entropy S_{JT} is equal to the statistical entropy

$$S_{CFT_1} = 2\pi\sqrt{\frac{cL_0}{6}} = 2\pi Q\sqrt{2Q\Delta M} \quad (26)$$

with the central charge $c = 12Q^3\alpha/\beta$ and the Virasoro generator $L_0^R = kQ\alpha\beta$. This is a realization of the $\text{AdS}_2/\text{CFT}_1$ correspondence in the near-horizon region.

VI. DISCUSSIONS

For the Schwarzschild black hole with $Q = 0$, we have no point of $\phi = \phi_0, \phi_m$. Here we have $T_H = 1/8\pi M$, $C = -8\pi M^2$, $F = M/2$ in the Schwarzschild phase. Similarly, we have $J = \sqrt{\phi}$, $V = 1/2\sqrt{\phi}$, $V' = -1/4\phi^{3/2}$ in the 2D dilaton gravity approach. Considering $\sqrt{\phi} = M$ in the Schwarzschild phase, we confirm that the thermodynamic duality between the Schwarzschild black hole and 2D dilaton gravity holds. Also it is clear that the duality is manifested by the conformal transformation (7) after integration over S^2 because it enforces the dilaton potential to be Eq. (9).

In conclusion, we have completely described the thermodynamic duality between the RN black hole and the 2D dilaton gravity. The key ingredient is to fully use the dilaton potential induced by the dimensional reduction and the conformal transformation. Even though, for simplicity, we have considered the RN black hole and the Schwarzschild black hole with $Q = 0$, the dilaton potential is available for all spherically symmetric black holes in four dimensions. Hence, our approach simplifies the study on the black hole significantly such

that the 4D black hole thermodynamics can be completely described by the 2D dilaton potential.

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- [1] J. D. Bekenstein, *Lett. Nuovo Cimento* **4**, 737 (1972).
 - [2] J. D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
 - [3] J. D. Bekenstein, *Phys. Rev. D* **9**, 3292 (1974).
 - [4] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
 - [5] J. M. Bardeen, B. Carter and S. W. Hawking, *Commun. Math. Phys.* **31**, 161 (1973).
 - [6] E. Witten, *Phys. Rev. D* **44**, 314 (1991).
 - [7] G. Mandal, A. M. Sengupta, S. R. Wadia, *Mod. Phys. Lett. A* **6**, 1685 (1991).
 - [8] C. G. Callan, S. B. Giddings, J. A. Harvey, A. Strominger, *Phys. Rev. D* **45**, R1005 (1992).
 - [9] J. G. Russo, L. Susskind, L. Thorlacius, *Phys. Lett. B* **292**, 13 (1992).
 - [10] V. P. Frolov, *Phys. Rev. D* **46**, 5383 (1992).
 - [11] T. M. Fiola, J. Preskill, A. Strominger, S. P. Trivedi, *Phys. Rev. D* **50**, 3987 (1994).
 - [12] R. Jackiw, in *Quantum Theory of Gravity*, ed. S. M. Christensen (Hilger, Bristol, 1984).
 - [13] C. Teitelboim, in *Quantum Theory of Gravity*, ed. S. M. Christensen (Hilger, Bristol, 1984).
 - [14] M. Henneaux, *Phys. Rev. Lett.* **54**, 959 (1985).
 - [15] D. A. Lowe and A. Strominger, *Phys. Rev. Lett.* **73**, 1468 (1994).
 - [16] R. B. Mann, D. Robbins, and T. Ohta, *Phys. Rev. Lett.* **82**, 3738 (1999).
 - [17] D. Christensen, R. B. Mann, *Class. Quantum Grav.* **9**, 1769 (1992).
 - [18] A. Achúcarro, M. E. Ortiz, *Phys. Rev. D* **48**, 3600 (1993).
 - [19] J. P. S. Lemos, P. M. Sá, *Phys. Rev. D* **49**, 2897 (1994).

- [20] M. Cadoni, S. Mignemi, *Phys. Rev. D* **51**, 4319 (1995).
- [21] A. Kumar, K. Ray, *Phys. Lett. B* **351**, 431 (1995).
- [22] M. Cadoni, *Class. Quantum Grav.* **22**, 409 (2005).
- [23] A. Fabbri, D. J. Navarro, and J. Navarro-Salas, *Nucl. Phys. B* **595**, 381 (2001).
- [24] D. Grumiller and R. McNees, *JHEP* **0704**, 074 (2007).
- [25] D. Grumiller, W. Kummer, D. V. Vassilevich, *Phys. Rept.* **369**, 327 (2002).
- [26] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Phys. Rev. D* **60**, 064018 (1999).
- [27] J. Gegenberg, G. Kunstatter and D. Louis-Martinez, *Phys. Rev. D* **51**, 1781 (1995).
- [28] J. Cruz, A. Fabbri, D. J. Navarro, and J. Navarro-Salas, *Phys. Rev. D* **61**, 024011 (2000).
- [29] Y. S. Myung, Y. W. Kim and Y. J. Park, *Phys. Rev. D* **76**, 104045 (2007).
- [30] Y. S. Myung, *Phys. Lett. B* **624**, 297 (2005).
- [31] J. Navarro-Salas and P. Navarro, *Nucl. Phys. B* **579**, 250 (2000).